SOME CALCULATIONS WITH TRANSFER IN SYMPLECTIC COBORDISM

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Abstract. This note provides some computations with transfer associated to Spin-bundles. One aspect is to revise the proof of the main result of [2] which says that all fourfold products of the Ray classes are zero in symplectic cobordism.

1. Introduction

The Ray classes [7] $\phi_i \in MSp_{8i-3}$ are indecomposable torsion elements of order two in symplectic bordism. ϕ_i arise from the expansion of Conner-Floyd symplectic Pontryagin class

$$pf_1((\eta^1 - \mathbb{R}) \otimes_{\mathbb{R}} (\zeta - \mathbb{H})) = s \sum_{i \geq 1} \theta_i pf_1^i(\zeta)$$

in $MSp^4(S^4 \wedge BSp(1))$, where s is the generator of $MSp^1(S^1) = \mathbb{Z}$, $\eta^1 \to S^1$ is the non-trivial real line bundle and $\zeta \to BSP(1)$ is the canonical $Sp(1)$ bundle. The notation

$$\theta_{2i} = \phi_i$$

is used in the literature because $\theta_{2i+1} = 0$, for $i > 1$ [8].

In [2] (Prop. 4.1) we proved the following

Theorem 1.1. *i) All fourfold products of the Ray classes $\phi_i\phi_j\phi_k\phi_l$ are zero;*
ii) The images of all double products $\phi_i\phi_j$ in self-conjugate cobordism are zero.

In this note we revise the proof of Theorem 1.1 as follows. In [2] Remark 1.11, Lemma 1.12, the proof of Proposition 1, (1.1) and (1.2), case $m = 5$ don't seem to be true. All these points are inherited from the references and are used to derive the proof of Proposition 1 of [2], which we cover in Section 4 by applying the calculations with transfer in symplectic cobordism in Section 3. For the reader's convenience, in Section 4 we briefly recall the proof of Theorem 1.1 by pointing to the sequence of necessary propositions of [2].

2. Preliminaries

Recall that the groups $Spin(n)$ and $Pin(n)$ [1] operate on $\mathbb{R}^n$ by vector representation.

We will use an octonionic representation of Clifford algebra $Cl(8, 0)$. For details we refer to [9].

One has the isomorphism of Clifford algebras

$$Cl^0(q + 1, p) \simeq Cl(p, q) \simeq Cl^0(p, q + 1) \tag{2.1}$$

obtained from extending

$$e_1e_{k+1} \leftarrow e_k \to e_ke_{n+1}, \ \ (1 \leq k \leq n).$$

The right isomorphism induces the inclusion of $Pin(n) = Pin^0(n) + Pin^1(n)$ in $Spin(n + 1)$, where $Pin^0(n) = Spin(n)$.

Let $\{e_0, e_1, \cdots e_7\}$ be an orthonormal basis of $V = \mathbb{R}^8$. Note that we choose induces ranging from 0 to 7. The octonionic algebra $\mathbb{Q}$ is assumed to be given with basis $\{i_0, i_1, \cdots, i_7\}$ obeying the multiplication table

$$i_0 = 1, \ \ i_k^2 = -1, \ \ i_ki_l = i_m = -i_li_k, \ \ 1 \leq k \leq 7, \quad \text{and cyclic for}$$
$$(k, l, m) \in P = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 6, 4), (2, 5, 7), (3, 4, 7), (3, 5, 6)\}.$$

One can identify V with $\mathbb{O}$ as vector spaces by $\sum x^k e_k \to \sum x^k i_k$.

An octonionic representation $Cl(8,0) \to M_2(\mathbb{O})$ is given by

$$\Gamma_k = \gamma_8(e_k) = \begin{pmatrix} 0 & i_k \\ i_k^* & 0 \end{pmatrix}, \ 0 \le k \le 7.$$

$$\Rightarrow \gamma_8(x) = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}, \ x \in V.$$

The carrier space of the representation is understood to be $\mathbb{O}^2$, i.e., the set of columns of two octonions, with γ_8 acting on it by left multiplication.

Restricting the representation $Cl(8,0)$ to $Cl^0(8,0) = Cl^0(0,8)$ produces a faithful representation with the generators

$$\Gamma_0\Gamma_k = \gamma_8(e_k) = \begin{pmatrix} i_k & 0 \\ 0 & -i_k \end{pmatrix}, \ 1 \le k \le 7.$$

So $Cl^0(0,8)$ is represented by diagonal matrices. This representation decomposes into two irreducible representations given by the two elements on the diagonal. By the isomorphism $Cl^0(8,0) = Cl(0,7)$ these two are also irreducible representations $Cl(0,7) \to \mathbb{O}$.

Let

$$\gamma_7 : Cl(0,7)) \to \mathbb{O}$$

be the irreducible representation given by the generators

$$\gamma_7(e_k) = i_k, \ \ 1 \le k \le 7, \tag{2.2}$$

$$\Leftrightarrow \ \gamma_7(x) = Imx, \ \ x \in Im\mathbb{O}, \tag{2.3}$$

which act by successive left multiplication on the carrier space $\mathbb{O}$.

Orthogonal transformations are generated by unit vectors $u \in Im\mathbb{O}$:

$$x' = \phi_{(\gamma_7(u))}(x) = uxu^{-1} = -uxu, \ \ x \in \mathbb{O}.$$

By the isomorphism $Cl^0(0,7) \simeq Cl(0,6)$, we obtain a faithful and irreducible representation

$$\gamma_6 : Cl(0,6) \to \mathbb{O} :$$

$$\gamma_6(e_k) = i_k i_7, \ \ 1 \le k \le 6, \tag{2.4}$$

$$\gamma_6(u) = ui_7, \ \ u \in \mathbb{R}^6. \tag{2.5}$$

Orthogonal transformations are generated by

$$x' = i_k(i_7 x i_7)i_k = i_k((i_7 x i_7)i_k), \ \ x \in \mathbb{O}.$$

Using (2.1) for $n = 6,5,4,3$ one can see how $Spin(n)$ operates in R^n identified with the imaginary subspace of $\mathbb{O}$ with vanishing $n+1, \cdots, 7$-components.

3. Spin bundles

The following bundles induced by the inclusion of groups

$$i_m : BSpin(m) \to BSpin(m+1), \tag{3.1}$$

$$j_m : BPin(m) \to BSpin(m+1) \tag{3.2}$$

can be considered as the sphere bundle and the projective bundle of the universal $Spin(m)$ bundle

$$\xi^m \to BSpin(m) \tag{3.3}$$

respectively.

Denote the sphere bundle and the projective bundle of a vector bundle ξ by $S(\xi)$ and $P(\xi)$ respectively. In particular we have

$$S(\xi^m) = BSpin(m), \ \ P(\xi^m) = BPin(m)$$

and the pullback bundles induced by inclusion $Spin(m) \hookrightarrow Spin(m+1)$,

$$S(\xi^m \oplus 1) \to BSpin(m), \tag{3.4}$$

$$P(\xi^m \oplus 1) \to BSpin(m). \tag{3.5}$$

Lemma 3.1. *Let $\xi^7 \to BSpin(7)$ be the universal $Spin(7)$ bundle as above and let*

$$\xi = 1 + \xi^7.$$

Let $\pi : P(\xi) \to BSpin(7)$ be the projective bundle of ξ:

$$P(\xi) = ESpin(7) \times_{Spin(7)} RP^7$$

and let $\mathcal{T}_F(\xi)$ be the tangent bundle along the fibers of π. Then

$$\mathcal{T}_F(\xi) = \pi^*(\xi^7).$$

Proof. Clearly $\phi_{\gamma_7(u)}$ induces the action of $Spin(7)$ on $\mathbb{O}$, also on the real projective space

$$RP^7 = \{\{\pm x\} \big| x \in \mathbb{O}, |x| = 1\}$$

and on the tangent bundle of RP^7:

$$\tau_F = RP^7 \times R^7 = \{\pm(x, v(x)) \big| v(x) = t_1 i_1 x + \cdots + t_7 i_7 x,\ t_1, \cdots, t_7 \in \mathbb{R}\}.$$

$Spin(7)$ acts trivially of on the line in $\mathbb{R}^8 = \mathbb{O}$ directed by i_0. The action on pure octonions defines the universal $Spin(7)$ bundle ξ^7.

This defines

$$\mathcal{T}_F(\xi) = ESpin(7) \times_{Spin(7)} \tau_F$$

and the bundle map

$$ESpin(7) \times_{Spin(7)} \tau_F \to ESpin(7) \times_{Spin(7)} R^7 = \xi^7,$$

which classifies $\pi^*(\xi^7)$. □

It is well known that RP^7 is paralelizable, i.e., admits 7 linearly independent tangent vector fields $(\{\pm p, \pm pi_1\}), \cdots, (\{\pm p, \pm pi_7\})$, where i_k are the octonionic units.

Lemma 3.2. *There are $7-k$ number $Spin(k)$-equivariant linearly independent tangent vector fields on RP^7, namely $(\{\pm p, \pm pi_{k+1}\}), \cdots, (\{\pm p, \pm pi_7\})$, where $k = 2, \cdots, 6$.*

Proof. Let $k = 6$ and let us check that the vector field $(\{\pm p, \pm pi_7\})$ on RP^7 is invariant under action of $Spin(6) \subset Cl_6^0$: Using Moufang identities

$$(xyx)z = x(y(xz)); \tag{3.6}$$

$$z(xyx) = ((zx)y)x; \tag{3.7}$$

$$x(yz)x = (xy)(zx) \tag{3.8}$$

one has for $\{\pm p\} = \pm\{t_0 i_0 + t_1 i_1 + \cdots t_7 i_7\}$

$$\begin{aligned}
\phi_{\gamma_6(e_k)}(p) &= i_k((i_7 p i_7) i_k) = i_k(i_7(t_0 i_0 + t_1 i_1 + \cdots t_7 i_7) i_7) i_k) \\
&= i_k(\sum_{n \neq 0,7} (t_n i_n - t_0 i_0 - t_7 i_7) i_k = \sum_{n \neq k,7} t_n i_n - t_k i_k - t_7 i_7.
\end{aligned}$$

$$\Rightarrow \phi_{\gamma_6(e_j e_k)}(p) = \sum_{n \neq j,k} t_n i_n - t_j i_j - t_k i_k;$$

$$\Rightarrow \phi_{\gamma_6(e_k)}(i_7) = -i_7;$$

$$\Rightarrow \phi_{\gamma_6(e_j e_k)}(i_7) = i_7;$$

$$\begin{aligned}
\phi_{\gamma_6(e_k)}(pi_7) &= (i_k i_7)(pi_7)(i_7 i_k) \\
&= ((i_k i_7)p)(i_7(i_7 i_k)) && \text{by (3.8), } x = i_k i_7,\ y = p,\ z = i_7 \\
&= -((i_k i_7)p)i_k && \text{as } i_7^2 = -1 \\
&= ((i_7 i_k)p)i_k && \text{as } i_7 i_k = -i_k i_7 \\
&= i_7(i_k p i_k) && \text{by (3.7), } x = i_k,\ p = y,\ z = i_7, \\
&= i_7(i_k(t_0 i_0 + t_1 i_1 + \cdots t_7 i_7)i_k) \\
&= i_7(\sum_{n\neq 0,k} t_n i_n - t_0 i_0 - t_k i_k) \\
&= (-\sum_{n\neq k,7} t_n i_n + t_k i_k + t_7 i_7)i_7 \\
\Rightarrow \phi_{\gamma_6(e_j e_k)}(pi_7) &= (\sum_{n\neq j,k} t_n i_n - t_j i_j - t_k i_k)i_7. \\
\Rightarrow \phi_{\gamma_6(e_j e_k)}(pi_7) &= \phi_{\gamma_6(e_j e_k)}(p)i_7.
\end{aligned}$$

Similarly for $k = 5, 4, 3$.

□

Corollary 3.3. Let ξ^k be the universal $Spin(k)$ bundle, $k = 2, \cdots, 6$. Then the tangent bundle along the fibers RP^7 of the projective bundle

$$\tilde{\pi} : P(8 - k + \xi^k) \to BSpin(k),$$

admits $(7 - k)$ linearly independent sections

$$\mathcal{T}_F(\xi^k + 8 - k) = \tilde{\pi}^*(\xi^k) + 7 - k.$$

Proof. Apply Lemma 3.1. For the standard inclusion $i_k : BSpin(k) \to BSpin(7)$ one has

$$i_k^*(\xi^7) = \xi^k + 7 - k,$$

therefore

$$\begin{aligned}
&\mathcal{T}_F(i_k^*(\xi^7 + 1)) = i_k^*(\tilde{\pi}^*(\xi^7)) \\
\Leftrightarrow\ &\mathcal{T}_F(\xi^k + 8 - k)) = \tilde{\pi}^*(\xi^k) + 7 - k.
\end{aligned}$$

Alternatively one can apply Lemma 3.2 to define $(7 - k)$-sections of the tangent bundle along the fibers of $\mathcal{T}_F(i_k^*(\xi))$.

□

Let $Tr^*(i_{m-1})$ and $Tr^*(j_{m-1})$ be the transfer homomorphism of (3.1) and (3.2) respectively. Then by naturality of the transfer $i_m^* Tr^*(j_m)$ is the transfer homomorphism of (3.5).

Lemma 3.4. *Let $2 \leq m \leq 7$. The transfer homomorphism of* (3.5) *is the sum of three components,*

$$i_m^* Tr^*(j_m) = Tr(j_{m-1})^* - Tr(i_{m-1})^* + Id.$$

This corresponds to the endpoints and the interior of the orbit type manifold

$$Spin(m)\big|Spin(m + 1)\big|Pin(m)$$

which is the line segment. The corresponding isotropy groups are: $Spin(m)$ at one endpoint, $Pin(m - 1)$ at another endpoint, and $Spin(m - 1)$ for the points in the interior.

Proof. Lemma 3.4 coincides with Lemma 1.9 and Lemma 1.10 of [2] for $m = 4$ and $m = 3$ respectively. However for all cases it is convenient to use the octonionic representation of Clifford algebras in Section 2.

By naturality of the transfer map $i_m^* Tr(j_m)^*$ coincides with transfer homomorphism of (3.5).

Let $m = 7$. We consider RP^7 as $S^7_+ = S^7 \cap \{x_0 \geq 0\}$ with identified antipodal points in $S^6 = S^7 \cap \{x_0 = 0\}$. Parametrize S^7_+ as

$$v = \cos t \cdot i_0 + \sin t \cdot x, \ \ x \in S^6 \subset Im\mathbb{O}, \ \ 0 \leq t \leq \pi/2.$$

Then as above i_0 is invariant under action of $Spin(7)$ and we have

$$v' = i_k(i_7(\cos t \cdot i_0 + \sin t \cdot x)i_7)i_k = \cos t \cdot i_0 + \sin t \cdot i_k(i_7 x i_7)i_k.$$

So the orbit space of the action of $Spin(7)$ on RP^7 is the line segment $[0, \pi/2]$: we have three types of orbits: the endpoint $t = 0$ corresponds to the pole e_0, with the isotropy group $Spin(7)$. The endpoint $t = \pi/2$ corresponds to the orbit $RP^6 = \{\pm x\}$, its points have the isotropy groups conjugate to $Pin(6)$, the isotropy group of $\{\pm i_7\}$. Each point $t \in (0, \pi/2)$ corresponds to the orbit $\cos t \cdot e_0 + \sin t \cdot x$, the sphere, consisting of points with the isotropy group conjugate to $Spin(6)$.

Now let $m = 6$ and consider RP^6 as $S^6_+ = S(Im\mathbb{O}) \cap \{x_7 \geq 0\}$ with identified antipodal points in $S^5 = S^6 \cap \{x_7 = 0\}$. Parametrize S^6_+ as

$$v = \cos t \cdot i_7 + \sin t \cdot x, \ \ x \in S^5, \ \ 0 \leq t \leq \pi/2.$$

As above i_7 is invariant under action of $\imath_j i_k \in Spin(6)$ and we have

$$v' = i_j i_k(\cos t \cdot i_7 + \sin t \cdot x)i_k i_j = \cos t \cdot i_7 + \sin t \cdot i_j(i_k x i_k)i_j.$$

The orbit space of the action of $Spin(6)$ on RP^6 is the line segment $[0, \pi/2]$ again: we have three types of orbits: the endpoint $t = 0$ corresponds to the pole e_7, with the isotropy group $Spin(6)$. The endpoint $t = \pi/2$ corresponds to the orbit $RP^5 = \{\pm x\}$, its points have the isotropy groups conjugate to $Pin(5)$, the isotropy group of $\{\pm i_6\}$. Each point $t \in (0, \pi/2)$ corresponds to the orbit $\cos t \cdot e_7 + \sin t \cdot x$, the sphere, consisting of points with the isotropy group conjugate to $Spin(5)$.

The proof for $m = 5, 4, 3$ is identical and is left to the reader.

□

Consider again the bundles (3.1) and (3.2). Let $\lambda \to P(\xi^{m-1})$ be the canonical real line bundle. λ splits off the bundle $j^*_{m-1}(\xi^m)$ as the canonical direct summand. Let f_{m-1} be the classifying map of λ.

Lemma 3.5. *One has for the composition of the transfer map Tr_m followed by the classifying map f_m is zero in symplectic cobordism*

$$\begin{array}{llr} i) & i_m^* Tr^*(j_m) f_m^* = Tr^*(j_{m-1}) f^*_{m-1}, & 2 \leq m \leq 7; \\ ii) & Tr^*(j_m) f_m^* = 0, & 2 \leq m \leq 6. \end{array}$$

Proof. For i) apply Lemma 3.4. The pullback bundle $i_m^* Tr^*(j_m)$ coincides with (3.5). Then $BSpin(m)$ is simply connected for $m \geq 2$: this is the consequence of the exact sequence of homotopy groups of the fibration $Spin(m) \to ESpin(m) \to BSpin(m)$ and the fact that $Spin(m)$ is path connected for $m \geq 2$ [1]. So that only the first component of the transfer homomorphism in Lemma 3.4 is relevant.

For ii) let $m = 6$. By Corollary 3.3 transfer map $i_6^* Tr_7$ of

$$i_6^*(j_7) = P(\xi^6 \oplus \mathbb{R}^2)$$

is trivial as its fiber RP^7 admits $Spin(6)$-equivariant vector field (p, pi_7) and therefore the bundle along the fibers of

$$P(\xi^6 \oplus \mathbb{R}^2) \to BSpim(6)$$

admits a section. Now apply again Lemma 3.4 for $m = 7$ and then $m = 6$ to complete the proof of ii) for $m = 6$.

Then for $m < 6$ the transfer homomorphism of

$$P(\xi^m \oplus \mathbb{R}^{8-m}),\ \ 2 \leq m < 6$$

is trivial as the pullback of $i_6^* Tr_7$ by i_m. Apply again Lemma 3.4 for $m+1$ and m to complete the proof of ii).

□

Recall from [6] that any $Spin(5)$-bundle is MSp-orientable. So is $\xi^m \oplus \mathbb{R}^{8-m}$ for $m \leq 5$. Therefore Lemma 3.1 implies that $\mathcal{T}_F(P(\xi^m \oplus \mathbb{R}^{8-m}))$ is MSp-orientable. Recall from [3] that in our situation the transfer homomorphism is expressed by Boardman's "Umkern" map which is zero because of zero Euler class.

4. Proof of Theorem 1.1

Here we follow the notations of [2]. The proof of Theorem 1.1 is organized as follows.

The tensor square of the canonical $Sp(1)$-bundle $\zeta \to BSp(1)$ has a trivial summand

$$\zeta \otimes_H \zeta^* = \Lambda + 1,$$

where $\Lambda \to BSp(1)$ is the canonical $Spin(3)$-bundle.

Let N be the normalizer of the torus $S^1 = U(1)$ in $S^3 = Sp(1) = Spin(3)$. Clearly the bundle

$$p : BN \to BSp(1) = BSpin(3)$$

is the projective bundle of Λ. The quotient map $N/U(1) = Z/2$ induces the map

$$f : BN \to BZ/2,$$

the classifying map of the canonical real line bundle

$$\lambda \to BN,\ \ \lambda^{\otimes 2} = 1. \tag{4.1}$$

The pullback of Λ splits canonically over BN

$$p^*(\Lambda) = \lambda + \mu. \tag{4.2}$$

Lemma 3.5 case $m = 2$ implies

$$Tr^* f^* = 0 \tag{4.3}$$

in symplectic cobordism, where Tr is the transfer map of p.

Then it turns out [6], [2] p.4394, that Λ is MSp-orientable and the Thom class can be chosen in such a way that its restriction to the zero section is equal to

$$\tilde{e}(\Lambda) = \theta_1 + \sum_i \phi_i x^i,\ \ x = pf_1^i(\zeta). \tag{4.4}$$

Now let Λ_i be the pullback of Λ induced by projection on i-th factor $BSp(1)^4 \to BSp(1)$ and λ be as above. Then ([2], Lemma 4.5)

$$\lambda \otimes_{\mathbb{R}} \sum_1^4 \Lambda_i \to B\mathbb{Z}/2 \times BSp(1)^4 \text{ is } MSp\text{-orientable}, \tag{4.5}$$

$$\lambda \otimes_{\mathbb{R}} \sum_1^2 \Lambda_i \to B\mathbb{Z}/2 \times BSp(1)^2 \text{ is } SC\text{-orientable}. \tag{4.6}$$

Because of (4.2) and (4.1) the pullback of (4.5) over

$$(f, p) \times 1 : BN \times BSp(1)^3 \to B\mathbb{Z}/2 \times BSp(1)^4$$

has a trivial summand and therefore zero MSp-orientation Euler class.

Similarly for the pullback of (4.6) over

$$(f, p) \times 1 : BN \times BSp(1) \to B\mathbb{Z}/2 \times BSp(1)^2.$$

Thus ([2], Lemma 4.6) one has in $MSp^*(BN \times BSp(1)^3)$

$$0 = \prod_{s=1}^{4}(\theta_i + \sum_{r\geq 1}\phi_r x_s^{2r}) + \sum_{m,n,p,q\geq 0} f^*(\gamma_{mnpq})x_1^m x_2^n x_3^p x_4^q \tag{4.7}$$

$$= \sum_{i,j,k,l\geq 1}\phi_i\phi_j\phi_k\phi_l x_1^{2i}x_2^{2j}x_3^{2k}x_4^{2l} + \sum_{m,n,p,q\geq 0} f^*(\gamma_{mnpq})x_1^m x_2^n x_3^p x_4^q. \tag{4.8}$$

Here in (4.7) we use the relation $\theta_1\phi_i\phi_j = 0$ of [5].

Similarly one has in $SC^*(BN \times BSp(1))$

$$0 = \sum_{i,j\geq 1}\phi_i\phi_j x_1^{2i}x_2^{2j} + \sum_{m,n\geq 0} f^*(\gamma_{mn})x_1^m x_2^n. \tag{4.9}$$

Finally to complete the proof of Theorem 1.1 i) apply (4.8) and (4.3).

Similarly apply (4.9) and (4.3) to complete the proof of Theorem 1.1 ii).

References

[1] *M. F. Atiyah, R. Bott and A. Shapiro,* : *Clifford modules*, Topology, **3(suppl. 1)**(1964), 3-38.

[2] *M. Bakuradze,* : *The transfer and symplectic cobordism,* Trans. Amer. Math. Soc., **349, 11**(1997), 4385-4399.

[3] *J. C. Becker and D.H. Gottlieb* : *The transfer map and fibre bundles*, Topology **14** (1975), 1–12

[4] *M. Feshbach* : *The transfer and compact Lie groups*, Trans. Amer. Math. Soc. **251** (1979), 139–169

[5] *V. G. Gorbunov* : *Symplectic cobordism of projective spaces*, Math. Sbornik **181** (1990), 506–520

[6] *V. G Gorbunov and N. Ray* : *Orientation of $Spin(n)$ bundles and symplectic cobordism*, Publ. RIMS. Kyoto Univ. **28**, 1, (1992), 39–55.

[7] *N. Ray,* : *Indecomposables in $TorsMSp_*$,* Topology, **10**(1971), 261-270.

[8] *F. W. Roush,* : *On some torsion classes in symplectic bordism,* Preprint.

[9] *J. Schray, C. A. Manogue,* : *Octonionic representations of Clifford algebras and triality,*(1994), ArXiv:hep-th/9407179v1 .

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