On the existence of solution of the two stage variational problem

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Let $t_0 < \vartheta_0 < \vartheta_1 < t_1$ be given numbers ; let $W_p^1\left(W_k^1\right)$ be the space of absolutely continuous functions $x(t) \in R^n$, $t \in I_1 = [t_0, \vartheta_1]$ $\left(y(t) \in R^m, t \in I_2 = [\vartheta_0, t_1]\right)$ satisfying the condition $\dot{x}(\cdot) \in L_p$ $\left(\dot{y}(\cdot) \in L_k\right)$.

To each element $z = (\theta, x(\cdot), y(\cdot)) \in Z = I_3 \times W_p^1 \times W_k^1$ we assign the functional

$$J(z) = \int_{t_0}^{\theta} f(t, x(t), \dot{x}(t))dt + \int_{\theta}^{t_1} g(t, y(t), \dot{y}(t))dt \qquad (1)$$

with the following restrictions

$$x(t_0) = x_0 \;\; y(\theta) = G(\theta, x(\theta)), \;\; y(t_1) = y_1, \qquad (2)$$

where $I_3 = [\vartheta_1, \vartheta_2]$, the scalar function $f(t,x,u)\left(g(t,y,v)\right)$ is continuous on the set $I_1 \times R^n \times R^n$ $(I_2 \times R^m \times R^m)$ and continuously differentiable with respect to argument $(x,u)(y,v)$. Let the m-dimensional vector-valued function $G(t,x)$ be continuous on the set $I_3 \times R^n$, let $x_0 \in R^n$ and $y_1 \in R^m$ be given points.

Definition 1. An element $z \in Z$ is said to be admissible if $J(z)$ is finite and the conditions (2) hold.

Denote by Z_0 the set of admissible elements.

Definition 2. An element $z_0 \in Z_0$ is said to be optimal if for an arbitrary element $\forall z \in Z_0$ the inequality

$$J(z_0) \le J(z) \qquad (3)$$

holds.

The problem (1)-(3) is called the **two stage** problem of the calculus of variations and z_0 is called its solution. Let $n = k = 1$, $f = g$ and $G(t,x) = x + a$, where a is a given number, then the problem we will call Razmadze's problem [1].

Theorem 1. *There exists a solution of the problem (1)-(3) if the following conditions hold:*

1) *the set Z_0 is non-empty;*
2) *the functions $f(t,x,u)$ and $g(t,y,v)$ are convex with respect to arguments u and v, respectively. Moreover, $f(t,x,0^n) = g(t,y,0^m) = 0$, where $0^n\left(0^m\right)$ is zero of the space $R^n\left(R^m\right)$;*
3) *There exist numbers $\alpha > 0$, $\gamma > 0$, $\beta \in R$ and $\rho \in R$ such that the following growth conditions $f(t,x,u) \ge \alpha|u|^p + \beta$, $p > 1$; $g(t,y,v) \ge \gamma|v|^k + \rho$, $k > 1$ hold.*

The existence of solution of the classical variational problem was proved in [2] for the first time. The Theorem 1 we will call Tonelli's type existence theorem.

References

[1] Razmadze A. M. Sur les solutions discontinues dans le calcul des variations. *Math. Ann.,* **94**, (1925), 1-52.

[2] Tonelli L. Sur la semi continuity des integrales doubles du calcul des variations. *Acta Math.,* **53,** (1929), 325- 346.